MA552. Midterm exam.

This takehome exam due October 17, 2006

1. Let A denote an $n \times n$ matrix with elements belonging to a field K. Let P denote the characteristic polynomial of A and assume $P(0) \neq 0$.

   Show that A has an inverse A^{-1} and that the characteristic polynomial R of A^{-1} is defined by

   $$R(\lambda) = \frac{(-1)^n \lambda^n}{P(0)} P\left(\frac{1}{\lambda}\right)$$

   Hint: one may, for example, examine the determinant of $A(A^{-1} - \lambda I)$. If K is a subfield of $\mathbb{C}$, one may use a triangular matrix B that is similar to A.

2. Consider the matrix A:

   (a) $A = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

   (1) Find the characteristic values and the eigenvectors of A

   (2) Find a nonsingular matrix T such that the matrix $T^{-1}AT$ will be a diagonal matrix D. Find the matrix T^{-1} and check your calculations by taking the product $T^{-1}AT$, write down D.

3. Let A be 2×2 matrix with complex elements and with a double characteristic value λ. Show that there exists a matrix B similar to A (that is, $B = T^{-1}AT$) and equal to one of two matrices

   $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

   Find B^n

4. Given the matrix A:

$$A = \begin{pmatrix} 8 & -1 & -5 \\ -2 & 3 & 1 \\ 4 & -1 & -1 \end{pmatrix}$$

(1) Find the characteristic values and the eigenvectors of A.

(2) Find a basis such that the transformed matrix of A, which we denote by B, will be triangular and find the matrix B.

(3) Show that among all bases that satisfy the condition of (2), there exists at least one such that the matrix B has one and only one nonzero element off the principal diagonal and that this term is equal to 1. Find the corresponding matrix B_0.

(4) Find B_0^n. Show without carrying the calculations, how one can find A^n.